ON A QUESTION OF D. SHLYAKHTENKO

by

IONUT CHIFAN and ADRIAN IOANA[1]

Abstract. In this short note we construct two countable, infinite conjugacy class groups which admit free, ergodic, probability measure preserving orbit equivalent actions, but whose group von Neumann algebras are not (stably) isomorphic.

Introduction.

Two countable, discrete groups Γ and Λ are *orbit equivalent* if they admit free, ergodic, probability measure preserving actions which generate isomorphic equivalence relations. They are W^*-*equivalent* if their group von Neumann algebras are isomorphic. D. Shlyaktenko noticed that, for all known examples of orbit equivalent groups, their group von Neumann algebras are isomorphic and speculated that this might be the case in general. Subsequently, a number of people have also asked the question of whether orbit equivalence of groups implies W^*-equivalence ([Oz06],[Po09]). Motivated by this question, we prove:

Theorem. *There exist two countable, discrete, infinite conjugacy class groups* Γ *and* Λ *which are orbit equivalent but not* W^*-*equivalent. Moreover, the group von Neumann algebras* $L\Gamma$ *and* $L\Lambda$ *are not stably isomorphic.*

The construction of the groups Γ and Λ is based on the observation that being an infinite conjugacy class group is not an orbit equivalence invariant. Indeed, by Dye's theorem, $\Gamma_0 = S_\infty$, the group of finite permutations of $\mathbb{N}$, is orbit equivalent to $\Lambda_0 = \mathbb{Z}$ ([OW80]). This example already shows that there are orbit equivalent groups which are not W^*-equivalent. Further, notice that the map $\Gamma_0 \to \Gamma = (\Gamma_0 \times \mathbb{F}_2) \star \mathbb{Z}$ turns every pair (Γ_0, Λ_0) of orbit equivalent groups into a pair (Γ, Λ) of orbit equivalent, infinite conjugacy class groups. Finally, by applying the Kurosh type results for free product von Neumann algebras from [Oz05] we derive that the group von Neumann algebras of Γ and Λ are not stably isomorphic.

[1]The second author was supported by a Clay Research Fellowship

Proof of theorem.

Before proving the theorem, we recall the notion of stable isomorphism of II_1 factors. For a II_1 factor M and $0 < t \in \mathbb{R}$, the amplification M^t is defined as the isomorphism class of $p(\mathbb{M}_n(\mathbb{C}) \otimes M)p$, where $n > t$ is an integer and $p \in \mathbb{M}_n(\mathbb{C}) \otimes M$ is a projection of trace $\frac{t}{n}$. It is well known that this isomorphism class does not depend on the choices of n and p. Then two II_1 factors are called *stably* isomorphic if one of them is isomorphic with an amplification of the other.

Proof. Let Γ_0 and Λ_0 be two infinite amenable groups and assume that Γ_0 is infinite conjugacy class (ICC) while Λ_0 is abelian. By [OW80], Γ_0 and Λ_0 are orbit equivalent. Further, by [Ga05, Section 2.2], the ICC groups $\Gamma = (\Gamma_0 \times \mathbb{F}_2) \star \mathbb{Z}$ and $\Lambda = (\Lambda_0 \times \mathbb{F}_2) \star \mathbb{Z}$ are orbit equivalent. We claim that the group von Neumann algebras $M = L\Gamma$ and $N = L\Lambda$ are not stably isomorphic. Let $M_0 = L\Gamma_0$ and $N_0 = L\Lambda_0$ and note that $M = (M_0 \overline{\otimes} L\mathbb{F}_2) \star L\mathbb{Z}$ and $N = (N_0 \overline{\otimes} L\mathbb{F}_2) \star L\mathbb{Z}$.

If we suppose by contradiction that this is not the case, then we can find an isomorphism $\theta : M^t \to N$, for some $t > 0$. Since M_0 is a factor we can view M_0^t as a subfactor of M^t. The commutant of M_0^t in M^t is then equal to $L\mathbb{F}_2$ ([Po83]). Since the latter is a non-injective factor, by applying [Oz05, Theorem 3.3.] we deduce that there is a unitary $u \in N$ such that $u\theta((M_0 \overline{\otimes} L\mathbb{F}_2)^t)u^* \subset N_0 \overline{\otimes} L\mathbb{F}_2$. By replacing θ with $\mathrm{Ad}(u) \circ \theta$ we can therefore assume that $\theta((M_0 \overline{\otimes} L\mathbb{F}_2)^t) \subset N_0 \overline{\otimes} L\mathbb{F}_2$.

Since the center of N_0 is diffuse, we derive that the commutant of $(M_0 \overline{\otimes} L\mathbb{F}_2)^t$ in M^t is diffuse. However, by [Po83] the commutant of $(M_0 \overline{\otimes} L\mathbb{F}_2)^t$ in M^t is equal to the center of $(M_0 \overline{\otimes} L\mathbb{F}_2)^t$. As M_0 is a factor, this gives a contradiction. $\square$

Acknowledgment. We are grateful to Professors Sorin Popa and Yehuda Shalom for useful discussions and encouragement.

References

[Ga05] D. Gaboriau: *Examples of groups that are measure equivalent to the free group*, Ergodic Theory Dynam. Systems **25** (2005), no. 6, 1809-1827.

[OW80] D. Ornstein, B. Weiss: *Ergodic theory of amenable groups.* I. *The Rokhlin lemma.*, Bull. Amer. Math. Soc. (N.S.) **1** (1980), 161–164.

[Oz05] N.Ozawa: *A Kurosh type theorem for type II_1 factors*, Int. Math. Res. Not. (2006), Volume 2006, 1–21, Article ID97560

[Oz06] N. Ozawa: *Amenable Actions And Applications*, Proceeding of the ICM 2006, Vol. II, 1563–1580.

[Po83] S. Popa: *Orthogonal pairs of $\star$-subalgebras in finite von Neumann algebras*, J. Operator Theory **9** (1983), 253–268.

[Po09] S. Popa: *Revisiting some problems in W^*−rigidity*, available at

http://www.math.ucla.edu/∼popa/workshop0309/slidesPopa.pdf.

Math Dept, UCLA, LA, CA 90095-155505
E-mail address: ichifan@math.ucla.edu

Math Dept, UCLA, LA, CA 90095-155505
E-mail address: aioana@caltech.edu